

A LOWER BOUND FOR THE FIRST EIGENVALUE OF A NEGATIVELY CURVED MANIFOLD

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There has been much work in recent years on the relation of the low eigenvalues of a compact Riemannian manifold to the geometry of the manifold. For Riemann surfaces with positive genus, it was observed by P. Buser [1] that one can find a compact hyperbolic surface of fixed genus (hence fixed area) with arbitrarily small first eigenvalue (see [10] for more information on this problem). For hyperbolic manifolds of dimension larger than two, Mostow's theorem implies that the topology uniquely determines the geometry, so the above phenomenon for λ_1 is likely to be a two-dimensional phenomenon. In this note we show that this is the case. Precisely, let M^n be a compact Riemannian manifold with sectional curvature bounded between two negative constants. We show here that if $n \geq 3$, then $\lambda_1(M)$ has a lower bound depending only on the volume of M . Actually, for $n > 3$, Gromov [7] has shown that an upper bound on volume implies an upper bound on diameter (for negatively curved M). Using this result, a bound such as ours would follow from a general result of S. T. Yau [11]. For $n = 3$, the diameter is not bounded in terms of volume (see [2, 3.13]) so our result seems to be of most interest in this case. Buser [2] has observed that our dependence on the inverse square of the volume is best possible.

The case $n = 3$ of our theorem was announced in the Hawaii Symposium in 1979. In this note we give a simplified version, valid for all $n > 2$, of our original proof. We wish to thank P. Buser for pointing out reference [9] which is used in the proof of Lemma 1.

The main results

We will assume throughout that M^n is a compact n dimensional manifold. We state our main result.

Theorem. *If the sectional curvatures of M satisfy the inequality $-1 \leq K_M \leq -\kappa^2$ for some $\kappa \in (0, 1)$ and if $n \geq 3$, then the first eigenvalue $\lambda_1(M)$ satisfies*

$$\lambda_1(M) \geq \min \left\{ \frac{(n-1)^2 \kappa^2}{4}, \frac{\delta_n}{\text{Vol}^2(M)} \right\} \geq \frac{\delta'_{n,\kappa}}{\text{Vol}^2(M)},$$

where $\delta_n = 4^{-1} \omega_{n-1}^2 [\varepsilon_n e^{-\varepsilon_n(1-\kappa)}]^{2n-2}$, $\varepsilon_n = 4^{-(n+3)}$, $\omega_n =$ volume of the unit ball in R^n , and

$$\delta'_{n,\kappa} = (n-1)^2 4^{-1} \kappa^2 \omega_n^2 \varepsilon_n^{2n}.$$

We now introduce some terminology. Given a hypersurface Σ in M and a local orthonormal frame e_1, \dots, e_{n-1} tangent to Σ , the mean curvature vector is given by

$$H = \frac{1}{n-1} \sum_{i=1}^{n-1} (D_{e_i} e_i)^{\text{Nor}},$$

where D is the Levi-Civita connection on M , and $()^{\text{Nor}}$ means projection normal to Σ . We will need three preliminary lemmas. The first is an isoperimetric inequality.

Lemma 1. *Suppose Σ is a closed (possibly disconnected) hypersurface in M which bounds a region Ω in M . Suppose the mean curvature vector H points everywhere into Ω , and assume the inequalities*

$$|H| \geq 1, \quad \text{Ric}_M \geq -(n-1).$$

Then we have $\text{Vol}(\Sigma) \geq (n-1)\text{Vol}(\Omega)$.

Proof. This result follows from the paper of Heintze-Karcher [9]. The estimates of [9, p. 453] applied on one side of Σ give the inequality

$$\text{Vol}(\Omega) \leq \left(\int_0^\infty \left(\cosh r - \left(\min_\Sigma |H| \right) \sinh r \right)_+^{n-1} dr \right) \text{Vol}(\Sigma),$$

where $()_+$ indicates the positive part of a function. Using the fact that $|H| \geq 1$, we get immediately the conclusion of Lemma 1.

For a point $P \in M$, let $i(P)$ denote the injectivity radius of M at P . Our next lemma gives an estimate of $i(P)$ for points along a hypersurface in terms of the volume of the hypersurface.

Lemma 2. *Suppose M satisfies $K_M \leq -\kappa^2$ for some $\kappa \geq 0$, and let Σ be a hypersurface in M with mean curvature H satisfying $|H| \leq \Lambda$. Suppose also that $\text{Vol}(\Sigma) < \infty$ and $\mathcal{H}^{n-2}(\bar{\Sigma} \sim \Sigma) = 0$ where \mathcal{H}^s denotes Hausdorff s dimensional measure. Then for every point $P \in \bar{\Sigma}$ we have*

$$i(P) e^{-i(P)(\Lambda-\kappa)_+} \leq [\omega_{n-1}^{-1} \text{Vol}(\Sigma)]^{1/n-1},$$

where ω_n denotes the volume of the unit ball in R^n .

Proof. The proof is a modification of a well known monotonicity inequality for the area of a submanifold of R^n . We do the proof assuming that Σ is closed since an easy cutoff argument can then be used to prove the general case. By standard comparison theorems, if r denotes the distance function to a point, $P \in M$, then we have the Hessian comparison

$$\frac{1}{2}D_{x,x}r^2 \geq (1 + \kappa r) |x|^2,$$

provided $r < i(P)$. Restricting this inequality to Σ and taking the trace we have

$$\frac{1}{2}\Delta_{\Sigma}r^2 \geq (n - 1)(1 - r(\Lambda - \kappa)_+).$$

Integrating this inequality over $\Sigma_{\tau} = \Sigma \cap B_{\tau}(P)$ and applying Stokes theorem we get

$$\tau \int_{\partial \Sigma_{\tau}} |\nabla r| \geq (n - 1)(1 - \tau(\Lambda - \kappa)_+) \text{Vol}(\Sigma_{\tau}),$$

where ∇ is the connection on Σ . Since for any regular value τ of $r|_{\Sigma}$ we have

$$\frac{d}{d\tau} \text{Vol}(\Sigma_{\tau}) = \int_{\partial \Sigma_{\tau}} |\nabla r|^{-1},$$

and since $|\nabla r| \leq 1$ on Σ , we get the differential inequality

$$\tau \frac{d}{d\tau} \text{Vol}(\Sigma_{\tau}) \geq (n - 1)(1 - \tau(\Lambda - \kappa)_+) \text{Vol}(\Sigma_{\tau}).$$

Integrating from ϵ to $i(P)$ we have

$$[\epsilon e^{-\epsilon(\Lambda - \kappa)_+}]^{1-n} \text{Vol}(\Sigma_{\epsilon}) \leq [i(P)e^{-i(P)(\Lambda - \kappa)_+}]^{1-n} \text{Vol}(\Sigma).$$

Letting $\epsilon \downarrow 0$ then gives the conclusion of Lemma 2.

The third preliminary lemma we need is a version of the Margulis lemma.

Lemma 3. *Suppose $-1 \leq K_M < 0$, and define a set Θ by $\Theta = \{P \in M: i(P) < \epsilon_n := 4^{-(n+3)}\}$. The set Θ is an open set having finitely many components $\Theta_1, \dots, \Theta_j$. Each component Θ_i is a neighborhood of a simple closed geodesic Γ_i with length $|\Gamma_i| < 2 \cdot \epsilon_n$. Moreover, each Θ_i is topologically equivalent to $S^1 \times B^{n-1}$, and is star-shaped with respect to Γ_i in the sense that every point of Θ_i is connected to Γ_i by a unique geodesic arc lying within Θ_i and meeting Γ_i orthogonally.*

Proof. By a version of the Margulis lemma given by Buser-Karcher [3, 2.5.4] we have, under our hypotheses, that if α, β are loops at a point $q \in M$ which have lengths $|\alpha|, |\beta| \leq 2\epsilon_n$ then α, β generate a cyclic subgroup of $\pi_1(M, q)$. Lemma 3 can be derived from this result as follows. Let $P \in \Theta$ be a given point, and let \tilde{P} be a point in the universal cover \tilde{M} of M lying above P . Since $i(P) < \epsilon_n$, there is a deck transformation γ which translates \tilde{P} a distance less than $2\epsilon_n$. Because M has negative curvature, there is a unique geodesic σ which

is preserved by γ . Let $\langle g \rangle$ denote the cyclic group of deck transformations which preserve σ . For any $h \in \langle g \rangle$, the function $\delta_h(x) = d(x, hx)$ is a convex function on M which achieves its minimum value on σ . The set $\tilde{\Theta}_i$ defined by

$$\tilde{\Theta}_i = \{x \in \tilde{M} : \delta_h(x) < \varepsilon_n \text{ for some } h \in \langle g \rangle\}$$

is therefore a finite union of convex neighborhoods of σ . Hence $\tilde{\Theta}_i$ is star-shaped with respect to σ . Now if k is a deck transformation such that both x and $k(x)$ lie in Θ_i for some x , then for some integers r, s , g^r (resp. g^s) translates x (resp. $k(x)$) a distance less than $2\varepsilon_n$. But then both g^r and $k^{-1}g^s k$ translate x less than $2\varepsilon_n$ and hence we have $k^{-1}g^s k \in \langle g \rangle$. From this it follows that g, k generate a solvable subgroup of π , which is cyclic by Preissman's theorem and hence $k \in \langle g \rangle$. Therefore, the set $\tilde{\Theta}_i / \langle g \rangle = \Theta_i$ is a domain in M containing the original point P and is a component of Θ . This gives the conclusions of Lemma 3.

Proof of Theorem. To prove the theorem we will use the isoperimetric quantity $h(M)$ of Cheeger [4] defined by

$$h(M) = \inf \left\{ \frac{\text{Vol}(\Sigma^{n-1})}{\min\{V_1, V_2\}} \right\},$$

where the infimum is taken over all smooth embedded hypersurfaces Σ^{n-1} (not necessarily connected) which divide M into two components with volumes V_1, V_2 . In [4], Cheeger proved the inequality

$$(1) \quad \lambda_1(M) \geq \frac{1}{4} h^2(M).$$

Thus we concentrate our efforts on giving a lower bound on $h(M)$. We will use the following existence theorem from minimal surface theory (see [5, Chapter 5], [6])

Existence Theorem. *For any v with $0 < v \leq \frac{1}{2} \text{Vol}(M)$, there exist an open set $\Omega_v \subset M$ with $\text{Vol}(\Omega_v) = v$, and a smooth embedded hypersurface Σ_v with the property that $\bar{\Sigma}_v = \partial\Omega_v$, $\mathcal{H}^s(\bar{\Sigma}_v \sim \Sigma_v) = 0$ for $s > n - 8$, and Ω_v has the extremal property*

$$\text{Vol}(\Sigma_v) = \inf\{\text{Vol}(\partial\Omega) : \Omega \subseteq M \text{ with } \text{Vol}(\Omega) = v\}.$$

Moreover, the mean curvature vector H of Σ_v satisfies $|H| \equiv H_v$ for a constant $H_v \geq 0$ as well as the property that H points everywhere into or everywhere out of Ω_v .

From the extremal property of Σ_v it is clear that

$$(2) \quad h(M) \geq \inf\{v^{-1} \text{Vol}(\Sigma_v) : 0 < v \leq \frac{1}{2} \text{Vol}(M)\}.$$

We divide our proof into two cases. First, if $H_v \geq 1$ then Lemma 1 can be applied to give

$$(3) \quad v^{-1} \text{Vol}(\Sigma_v) \geq n - 1.$$

Note that one has to take some care in applying Lemma 1 because for large n , Σ_v may have singularities. By the observation of Gromov [8], a nearest point to $\bar{\Sigma}_v$ from any given point of $M \sim \bar{\Sigma}_v$ is always a regular point and hence the methods of [9] are applicable.

The remaining case is $H_v < 1$. Now if it were true that

$$(4) \quad \text{Vol}(\Sigma_v) \geq \omega_{n-1} [\varepsilon_n e^{-\varepsilon_n(1-\kappa)}]^{n-1},$$

where $\varepsilon_n = 4^{-(n+3)}$, then we would be finished in light of (1)–(4). Therefore we assume that (4) does not hold. Then from Lemma 2 we would have $i(P) < \varepsilon_n$ for every $p \in \bar{\Sigma}_v$; that is, we have $\bar{\Sigma}_v \subseteq \emptyset$ in the terminology of Lemma 3. Since $n > 2$, the set $M \sim \emptyset$ is connected. Let U_v be the component of $M \sim \bar{\Sigma}_v$ which contains $M \sim \emptyset$, and let $\Omega'_v = M \sim \bar{U}_v$ and $\Sigma'_v = \Sigma_v \cap \partial\Omega'_v$. By construction we have

$$(5) \quad v^{-1} \text{Vol}(\Sigma_v) \geq \text{Vol}(\Omega'_v)^{-1} \text{Vol}(\Sigma'_v)$$

(recall that $v \leq \frac{1}{2} \text{Vol}(M)$). Let Ω be a component of Ω'_v and let $\Sigma = \Sigma_v \cap \partial\Omega$. Then $\bar{\Omega} \subset \emptyset_i$ for some component \emptyset_i of \emptyset . Since \emptyset_i is the quotient by a cyclic group of a star-shaped neighborhood of a geodesic σ in \tilde{M} (see the proof of Lemma 3), the distance function to σ is a well defined function in \emptyset_i which we denote by ρ . By standard comparison methods we have $\Delta_M \rho \geq (n - 1)\kappa$ in \emptyset_i . Thus Stokes theorem applied in Ω gives

$$(n - 1)\kappa \text{Vol}(\Omega) \leq \text{Vol}(\Sigma).$$

Since any two components of Ω'_v have disjoint closures, we can sum these inequalities over all components of Ω'_v to conclude

$$(6) \quad \text{Vol}(\Omega'_v)^{-1} \text{Vol}(\Sigma'_v) \geq (n - 1)\kappa.$$

Combining (1)–(6) we have

$$\lambda_1(M) \geq \min \left\{ \frac{(n - 1)^2 \kappa^2}{4}, \frac{\delta_n}{\text{Vol}^2(M)} \right\},$$

where $\delta_n = 4^{-1} \omega_{n-1}^2 [\varepsilon_n e^{-\varepsilon_n(1-\kappa)}]^{2n-2}$, $\varepsilon_n = 4^{-(n+3)}$. The final inequality of the theorem follows from this because by Lemma 3 there is a point $P \in M$ with $i(P) \geq \varepsilon_n$ hence the volume satisfies

$$\text{Vol}(M) \geq \omega_n \varepsilon_n^n.$$

Thus we have

$$\lambda_1(M) \geq \frac{\delta'_{n,\kappa}}{\text{Vol}^2(M)},$$

where $\delta'_{n,\kappa} = 4^{-1}(n-1)^2\kappa^2\omega_n^2\varepsilon_n^{2n}$. (Note that for $n \geq 3$ we have $\delta'_{n,\kappa} \leq \delta_n$.) This completes the proof of the main theorem.

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